How Synthetic Division Works
or
The Madness Behind the Method

by

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Students often ask why synthetic division is done the way it is and if it can be done with divisors of degree higher than one. The purpose of this note is to answer both questions. I have presented the calculations in considerable detail so the reader can easily adapt them to classroom presentation.

Linear Divisors

Let's divide $P(x) = x^4 - x^3 + 1$ by $D(x) = x - 2$. The synthetic division is simple:

$$
\begin{array}{r|rrrr|r}
2 & 1 & -1 & 0 & 0 & 1 \\
 & & 2 & 2 & 4 & 8 \\
\hline
 & 1 & 1 & 2 & 4 & 9 \\
 & \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5}
\end{array}
$$

and so

$$x^4 - x^3 + 1 = (x^3 + x^2 + 2x + 4)(x - 2) + 9.$$

The problem is justifying the method to your students. The idea is simple: add and subtract. The coefficients of each stage numbered below correspond to the coefficients obtained above. Thus,

$$
\begin{aligned}
x^4 - x^3 + 1 &= \underset{\textcircled{1}}{x^3}(x-2) + x^3 \cdot 2 - x^3 + 1 \\
&= x^3(x-2) + \underset{\textcircled{2}}{x^3} + 1 \\
&= x^3(x-2) + x^2(x-2) + x^2 \cdot 2 + 1 \\
&= x^3(x-2) + x^2(x-2) + \underset{\textcircled{3}}{2x^2} + 1 \\
&= x^3(x-2) + x^2(x-2) + 2x(x-2) + 2x \cdot 2 + 1 \\
&= x^3(x-2) + x^2(x-2) + 2x(x-2) + \underset{\textcircled{4}}{4x} + 1 \\
&= x^3(x-2) + x^2(x-2) + 2x(x-2) + 4(x-2) + 4 \cdot 2 + 1 \\
&= x^3(x-2) + x^2(x-2) + 2x(x-2) + 4(x-2) + \underset{\textcircled{5}}{9} \\
x^4 - x^3 + 1 &= (x^3 + x^2 + 2x + 4)(x-2) + 9
\end{aligned}
$$

Higher Degree Divisors

Now suppose we want to divide $P(x) = x^4 - x^3 + 1$ by $D(x) = x^2 + x - 2$. The long division is

$$
\begin{array}{r|rrrrrl}
 & & & x^2 & -\ 2x & +\ 4 & \\
\hline
x^2 + x - 2 & x^4 & -\ x^3 & +\ 0x^2 & +\ 0x & +\ 1 & \\
 & x^4 & +\ x^3 & -\ 2x^2 & & & \\
\hline
 & & -\ 2x^3 & +\ 2x^2 & +\ 0x & & \textcircled{1} \\
 & & -\ 2x^3 & -\ 2x^2 & +\ 4x & & \\
\hline
 & & & 4x^2 & -\ 4x & +\ 1 & \textcircled{2} \\
 & & & 4x^2 & +\ 4x & -\ 8 & \\
\hline
 & & & & -\ 8x & +\ 9 & \textcircled{3}
\end{array}
$$

and so

$$x^4 - x^3 + 1 = (x^2 - 2x + 4)(x^2 + x - 2) - 8x + 9.$$

OK – so far so good. Now let's play the same add and subtract game as in our first example. The numbered rows above correspond the numbered terms below. Thus,

$$\begin{aligned}
x^4 - x^3 + 1 &= x^2(x^2 + x - 2) + x^2(-x + 2) - x^3 + 1 \\
&= x^2(x^2 + x - 2) \ \underset{\textcircled{1}}{\underline{-\ 2x^3 + 2x^2 + 0x}} + 1 \\
&= x^2(x^2 + x - 2) - 2x(x^2 + x - 2) - 2x(-x + 2) + 2x^2 + 1 \\
&= x^2(x^2 + x - 2) - x(x^2 - 2) \ \underset{\textcircled{2}}{\underline{+\ 4x^2 - 4x + 1}} \\
&= x^2(x^2 + x - 2) - x(x^2 + x - 2) + 4(x^2 + x - 2) + 4(-x + 2) - 4x + 1 \\
&= x^2(x^2 + x - 2) - x(x^2 + x - 2) + 4(x^2 + x - 2) \ \underset{\textcircled{3}}{\underline{-\ 8x + 9}} \\
x^4 - x^3 + 1 &= (x^2 - x + 4)(x^2 + x - 2) - 8x + 9.
\end{aligned}$$

Now let's write this as a synthetic division. The point is that since we are performing the replacement

$$x^2 = (x^2 + x - 2) - x + 2,$$

we start as follows:

$$\begin{array}{cc|ccccc}
-1 & 2 & 1 & -1 & 0 & 0 & 1 \\ \hline
\end{array}$$

where the first marked off list is the coefficients of $-x + 2$ and the rest is the list of coefficients of $P(x)$. The idea is to multiply each leading coefficient (doubly underlined below) by the two divisor terms and bring the resulting pair down one row and right one column. Add with a carry down from the right. Repeat as necessary. Thus,

$$\begin{array}{cc|cccccl}
-1 & 2 & \underline{\underline{1}} & -1 & 0 & 0 & 1 & \\
\cline{1-2}
 & & & -1 & 2 & & & \\
\cline{4-6}
 & & & \underline{\underline{-2}} & 2 & 0 & & \textcircled{1} \\
 & & & & 2 & -4 & & \\
\cline{5-7}
 & & & & \underline{\underline{4}} & -4 & 1 & \textcircled{2} \\
 & & & & & -4 & 8 & \\
\cline{6-7}
 & & & & & -8 & 9 & \textcircled{3}
\end{array}$$

The coefficients of the quotient are obtained from the doubly underlined numbers as $Q = x^2 - 2x + 4$ and the remainder is read off the bottom as $R = -8x + 9$. Clearly, it's more compact and less error-prone than long division.

Awesome!